

Chapter 21. Trigonometric Identities

Ex 21.1

Answer 5.

$$(i) (\sec \theta - \tan \theta)^2 = \frac{1 - \sin \theta}{1 + \sin \theta}$$

$$\begin{aligned} (\sec \theta - \tan \theta)^2 &= \left(\frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} \right)^2 \\ &= \left(\frac{1 - \sin \theta}{\cos \theta} \right)^2 = \frac{(1 - \sin \theta)^2}{\cos^2 \theta} \\ &= \frac{(1 - \sin \theta)^2}{1 - \sin^2 \theta} = \frac{(1 - \sin \theta)^2}{(1 - \sin \theta)(1 + \sin \theta)} \quad (\because 1 - \sin^2 \theta = \cos^2 \theta) \\ &= \frac{1 - \sin \theta}{1 + \sin \theta} \end{aligned}$$

$$\begin{aligned} (ii) \frac{1}{\sin A + \cos A} + \frac{1}{\sin A - \cos A} &= \frac{2 \sin A}{1 - 2 \cos^2 A} \\ \frac{1}{\sin A + \cos A} + \frac{1}{\sin A - \cos A} &= \frac{\sin A - \cos A + \sin A + \cos A}{\sin^2 A - \cos^2 A} \\ &= \frac{2 \sin A}{1 - \cos^2 A - \cos^2 A} = \frac{2 \sin A}{1 - 2 \cos^2 A} \end{aligned}$$

$$\begin{aligned} (iii) \frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A} &= \frac{2}{2 \sin^2 A - 1} \\ \frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A} &= \frac{(\sin A + \cos A)^2 + (\sin A - \cos A)^2}{(\sin A + \cos A)(\sin A - \cos A)} \\ &= \frac{\sin^2 A + \cos^2 A + 2 \sin A \cos A + \sin^2 A + \cos^2 A - 2 \sin A \cos A}{\sin^2 A - \cos^2 A} \\ &= \frac{2(\sin^2 A + \cos^2 A)}{\sin^2 A - \cos^2 A} \\ &= \frac{2}{\sin^2 A - \cos^2 A} \quad [\sin^2 A + \cos^2 A = 1] \\ &= \frac{2}{\sin^2 A - \cos^2 A} = \frac{2}{\sin^2 A - (1 - \sin^2 A)} \\ &\Rightarrow \frac{2}{2 \sin^2 A - 1} \end{aligned}$$

$$(iv) \tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B}$$

$$\begin{aligned} \text{L.H.S} &= \tan^2 A - \tan^2 B \\ &= \frac{\sin^2 A}{\cos^2 A} - \frac{\sin^2 B}{\cos^2 B} \\ &= \frac{\sin^2 A \cos^2 B - \cos^2 A \sin^2 B}{\cos^2 A \cos^2 B} \\ &= \frac{(1 - \cos^2 A) \cos^2 B - \cos^2 A (1 - \cos^2 B)}{\cos^2 A \cos^2 B} \\ &= \frac{\cos^2 B - \cos^2 A \cos^2 B - \cos^2 A + \cos^2 A \cos^2 B}{\cos^2 A \cos^2 B} \\ &= \frac{\cos^2 B - \cos^2 A}{\cos^2 A \cos^2 B} \\ &= \frac{(1 - \sin^2 B) - (1 - \sin^2 A)}{\cos^2 A \cos^2 B} \\ &= \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B} \\ \text{Hence } \tan^2 A - \tan^2 B &= \frac{\cos^2 B - \cos^2 A}{\cos^2 A \cos^2 B} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cos^2 B} \end{aligned}$$

$$(v) \frac{\cos A}{1 - \tan A} + \frac{\sin^2 A}{\sin A - \cos A} = \cos A + \sin A$$

$$\begin{aligned} & \frac{\cos A}{1 - \tan A} + \frac{\sin^2 A}{\sin A - \cos A} \\ &= \frac{\cos A}{1 - \frac{\sin A}{\cos A}} + \frac{\sin^2 A}{\sin A - \cos A} \\ &= \frac{\cos A}{\frac{\cos A - \sin A}{\cos A}} + \frac{\sin^2 A}{\sin A - \cos A} \\ &= \frac{\cos^2 A}{(\cos A - \sin A)} - \frac{\sin^2 A}{(\cos A - \sin A)} \\ &= \frac{(\cos^2 A - \sin^2 A)}{(\cos A - \sin A)} = \frac{(\cos A + \sin A)(\cos A - \sin A)}{(\cos A - \sin A)} \\ &= (\cos A + \sin A) \end{aligned}$$

$$(vi) (1 + \tan^2 A) + \left(1 + \frac{1}{\tan^2 A}\right) = \frac{1}{\sin^2 A - \sin^4 A}$$

$$\begin{aligned} & (1 + \tan^2 A) + \left(1 + \frac{1}{\tan^2 A}\right) \\ &= \left(1 + \frac{\sin^2 A}{\cos^2 A}\right) + \left(1 + \frac{1}{\frac{\sin^2 A}{\cos^2 A}}\right) \\ &= \left(\frac{\cos^2 A + \sin^2 A}{\cos^2 A}\right) + \left(\frac{\cos^2 A + \sin^2 A}{\sin^2 A}\right) \\ &= \frac{1}{1 - \sin^2 A} + \frac{1}{\sin^2 A} \quad (\because \cos^2 A + \sin^2 A = 1) \\ &= \frac{\sin^2 A + 1 - \sin^2 A}{\sin^2 A(1 - \sin^2 A)} = \frac{1}{\sin^2 A - \sin^4 A} \end{aligned}$$

$$(vii) \frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A} = 2$$

$$\begin{aligned} & \frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A} \\ &= \frac{(\cos^3 A + \sin^3 A)(\cos A - \sin A) + (\cos^3 A - \sin^3 A)(\cos A + \sin A)}{\cos^2 A - \sin^2 A} \\ &= \frac{\cos^4 A - \cos^3 A \sin A + \sin^3 A \cos A - \sin^4 A + \cos^4 A + \cos^3 A \sin A - \sin^3 A \cos A + \sin^4 A}{\cos^2 A - \sin^2 A} \\ &= \frac{2(\cos^4 A - \sin^4 A)}{\cos^2 A - \sin^2 A} = \frac{2(\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)}{(\cos^2 A - \sin^2 A)} = 2(\cos^2 A + \sin^2 A) \\ &= 2 \quad (\because \cos^2 A + \sin^2 A = 1) \end{aligned}$$

OR

$$\begin{aligned} & \frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A} \\ &= \frac{(\cos A + \sin A)(\cos^2 A + \sin^2 A - \cos A \sin A)}{(\cos A + \sin A)} + \frac{(\cos A - \sin A)(\cos^2 A + \sin^2 A + \cos A \sin A)}{(\cos A - \sin A)} \end{aligned}$$

$$(\because a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2))$$

$$= (\cos^2 A + \sin^2 A - \cos A \sin A) + (\cos^2 A + \sin^2 A + \cos A \sin A)$$

$$= 1 - \cos A \sin A + 1 + \cos A \sin A \quad (\because \cos^2 A + \sin^2 A = 1)$$

$$= 2$$

$$(viii) \left(\tan \theta + \frac{1}{\cos \theta}\right)^2 + \left(\tan \theta - \frac{1}{\cos \theta}\right)^2 = 2 \left(\frac{1 + \sin^2 \theta}{1 - \sin^2 \theta}\right)$$

$$\begin{aligned} & \left(\tan \theta + \frac{1}{\cos \theta}\right)^2 + \left(\tan \theta - \frac{1}{\cos \theta}\right)^2 \\ &= \left(\frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta}\right)^2 + \left(\frac{\sin \theta}{\cos \theta} - \frac{1}{\cos \theta}\right)^2 \\ &= \left(\frac{\sin \theta + 1}{\cos \theta}\right)^2 + \left(\frac{\sin \theta - 1}{\cos \theta}\right)^2 \\ &= \frac{(\sin \theta + 1)^2}{\cos^2 \theta} + \frac{(\sin \theta - 1)^2}{\cos^2 \theta} \\ &= \frac{(\sin \theta + 1)^2 + (\sin \theta - 1)^2}{\cos^2 \theta} \\ &= \frac{\sin^2 \theta + 1 + 2 \sin \theta + \sin^2 \theta + 1 - 2 \sin \theta}{1 - \sin^2 \theta} \\ &= \frac{2(1 + \sin^2 \theta)}{1 - \sin^2 \theta} \end{aligned}$$

$$(ix) \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$$

$$\begin{aligned} \text{L.H.S.} &= \frac{\sin A + \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A - \sin B} \\ &= \frac{(\sin A + \sin B)(\sin A - \sin B) + (\cos A + \cos B)(\cos A - \cos B)}{(\cos A + \cos B)(\sin A - \sin B)} \\ &= \frac{\sin^2 A - \sin^2 B + \cos^2 A - \cos^2 B}{(\cos A + \cos B)(\sin A - \sin B)} \\ &= \frac{(\sin^2 A + \cos^2 A) - (\sin^2 B + \cos^2 B)}{(\cos A + \cos B)(\sin A - \sin B)} \\ &= \frac{1 - 1}{(\cos A + \cos B)(\sin A - \sin B)} \\ &= \frac{0}{(\cos A + \cos B)(\sin A - \sin B)} \\ &= 0 \end{aligned}$$

$$\frac{\sin A + \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A - \sin B} = 0$$

Hence proved.

$$\begin{aligned} (x) \frac{1}{\cos A + \sin A - 1} + \frac{1}{\cos A + \sin A + 1} &= \operatorname{cosec} A + \sec A \\ \frac{1}{(\cos A + \sin A) - 1} + \frac{1}{(\cos A + \sin A) + 1} & \\ &= \frac{\cos A + \sin A + 1 + \cos A + \sin A - 1}{(\cos A + \sin A)^2 - 1} \\ &= \frac{2(\cos A + \sin A)}{\cos^2 A + \sin^2 A + 2 \cos A \sin A - 1} \\ &= \frac{2(\cos A + \sin A)}{1 + 2 \cos A \sin A - 1} = \frac{\cos A + \sin A}{\cos A \sin A} \\ &= \frac{\cos A}{\cos A \sin A} + \frac{\sin A}{\cos A \sin A} \\ &= \frac{1}{\sin A} + \frac{1}{\cos A} \\ &= \operatorname{cosec} A + \sec A \end{aligned}$$

$$(xi) \frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1} = \frac{\cos A + 1}{\sin A}$$

$$\begin{aligned} &\frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1} \\ &= \frac{\cot A + \operatorname{cosec} A - (\operatorname{cosec}^2 A - \cot^2 A)}{\cot A - \operatorname{cosec} A + 1} \quad [\operatorname{cosec}^2 A - \cot^2 A = 1] \\ &= \frac{\cot A + \operatorname{cosec} A - [(\operatorname{cosec} A - \cot A)(\operatorname{cosec} A + \cot A)]}{\cot A - \operatorname{cosec} A + 1} \\ &= \frac{\cot A + \operatorname{cosec} A [1 - \operatorname{cosec} A + \cot A]}{\cot A - \operatorname{cosec} A + 1} \\ &= \cot A + \operatorname{cosec} A \\ &= \frac{\cos A}{\sin A} + \frac{1}{\sin A} \\ &= \frac{1 + \cos A}{\sin A} \end{aligned}$$

$$(xii) \frac{\sec A - 1}{\sec A + 1} = \frac{\sin^2 A}{(1 + \cos A)^2}$$

$$\begin{aligned} &\frac{\sec A - 1}{\sec A + 1} \\ &= \frac{\frac{1}{\cos A} - 1}{\frac{1}{\cos A} + 1} = \frac{1 - \cos A}{1 + \cos A} \\ &= \frac{1 - \cos A}{1 + \cos A} \times \frac{1 + \cos A}{1 + \cos A} \\ &= \frac{1 - \cos^2 A}{(1 + \cos A)^2} \\ &= \frac{\sin^2 A}{(1 + \cos A)^2} \quad (\because 1 - \cos^2 A = \sin^2 A) \end{aligned}$$

Answer 6.

$$\begin{aligned}
 \text{(i)} \quad & (1 + \cot A)^2 + (1 - \cot A)^2 = 2 \operatorname{cosec}^2 A \\
 & (1 + \cot A)^2 + (1 - \cot A)^2 \\
 & = 1 + \cot^2 A + 2 \cot A + 1 + \cot^2 A - 2 \cot A \\
 & = 2 + 2 \cot^2 A = 2(1 + \cot^2 A) \\
 & = 2 \operatorname{cosec}^2 A
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \frac{\operatorname{cosec} \theta}{\tan \theta + \cot \theta} = \cos \theta \\
 & \frac{\operatorname{cosec} \theta}{\tan \theta + \cot \theta} \\
 & = \frac{\frac{1}{\sin \theta}}{\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}} \\
 & = \frac{\frac{1}{\sin \theta}}{\frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta \sin \theta}} = \frac{\frac{1}{\sin \theta}}{\frac{1}{\cos \theta \sin \theta}} \\
 & = \frac{1}{\sin \theta} \times \frac{\cos \theta \sin \theta}{1} = \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & (1 + \tan^2 \theta) \sin \theta \cos \theta = \tan \theta \\
 & (1 + \tan^2 \theta) \sin \theta \cos \theta \\
 & = \left(1 + \frac{\sin^2 \theta}{\cos^2 \theta}\right) \sin \theta \cos \theta \\
 & = \left(\frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta}\right) \sin \theta \cos \theta \\
 & = \frac{1}{\cos^2 \theta} \times \sin \theta \cos \theta \quad (\because \cos^2 \theta + \sin^2 \theta = 1) \\
 & = \frac{\sin \theta}{\cos \theta} = \tan \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \frac{1 + \sin \theta}{\operatorname{cosec} \theta - \cot \theta} - \frac{1 - \sin \theta}{\operatorname{cosec} \theta + \cot \theta} = 2(1 + \cot \theta) \\
 & \frac{1 + \sin \theta}{\operatorname{cosec} \theta - \cot \theta} - \frac{1 - \sin \theta}{\operatorname{cosec} \theta + \cot \theta} \\
 & = \frac{(1 + \sin \theta)(\operatorname{cosec} \theta + \cot \theta) - (1 - \sin \theta)(\operatorname{cosec} \theta - \cot \theta)}{\operatorname{cosec}^2 \theta - \cot^2 \theta} \\
 & = \frac{\operatorname{cosec} \theta + \cot \theta + 1 + \cos \theta - \operatorname{cosec} \theta + \cot \theta + 1 - \cos \theta}{1 + \cot^2 \theta - \cot^2 \theta} \quad (\because \operatorname{cosec}^2 \theta = 1 + \cot^2 \theta) \\
 & = 2 + 2 \cot \theta = 2(1 + \cot \theta)
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad & (1 + \cot A + \tan A)(\sin A - \cos A) = \frac{\sec A}{\operatorname{cosec}^2 A} - \frac{\operatorname{cosec} A}{\sec^2 A} \\
 & (1 + \cot A + \tan A)(\sin A - \cos A) \\
 & = \left(1 + \frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}\right)(\sin A - \cos A) \\
 & = \left(\frac{\sin A \cos A + \cos^2 A + \sin^2 A}{\sin A \cos A}\right)(\sin A - \cos A) \\
 & = \frac{(\sin^3 A - \cos^3 A)}{\sin A \cos A} \quad (\because (\sin^3 A - \cos^3 A) = (\sin A - \cos A)(\sin A \cos A + \cos^2 A + \sin^2 A)) \\
 & = \frac{\sin^3 A}{\sin A \cos A} - \frac{\cos^3 A}{\sin A \cos A} \\
 & = \frac{\sin^2 A}{\cos A} - \frac{\cos^2 A}{\sin A} = \frac{1}{\cos A} \times \sin^2 A - \frac{1}{\sin A} \times \cos^2 A \\
 & = \sec A \sin^2 A - \operatorname{cosec} A \cos^2 A \\
 & = \frac{\sec A}{\operatorname{cosec}^2 A} - \frac{\operatorname{cosec} A}{\sec^2 A}
 \end{aligned}$$

$$(vi) 2(\sin^6 \theta + \cos^6 \theta) - 3(\sin^4 \theta + \cos^4 \theta) + 1 = 0$$

$$\begin{aligned} & 2(\sin^6 \theta + \cos^6 \theta) - 3(\sin^4 \theta + \cos^4 \theta) + 1 \\ &= 2[(\sin^2 \theta)^3 + (\cos^2 \theta)^3] - 3(\sin^4 \theta + \cos^4 \theta) + 1 \\ &= 2[(\sin^2 \theta + \cos^2 \theta)\{(\sin^2 \theta)^2 + (\cos^2 \theta)^2 - \sin^2 \theta \cos^2 \theta\}] - 3(\sin^4 \theta + \cos^4 \theta) + 1 \\ &= 2\{(\sin^2 \theta)^2 + (\cos^2 \theta)^2 - \sin^2 \theta \cos^2 \theta\} - 3(\sin^4 \theta + \cos^4 \theta) + 1 \\ &= 2\sin^4 \theta + 2\cos^4 \theta - 2\sin^2 \theta \cos^2 \theta - 3\sin^4 \theta - 3\cos^4 \theta + 1 \\ &= -\sin^4 \theta - \cos^4 \theta - 2\sin^2 \theta \cos^2 \theta + 1 \\ &= -(\sin^4 \theta + \cos^4 \theta + 2\sin^2 \theta \cos^2 \theta) + 1 \\ &= -(\sin^2 \theta + \cos^2 \theta)^2 + 1 = -1 + 1 = 0 \end{aligned}$$

$$(vii) \sin^8 \theta - \cos^8 \theta = (\sin^2 \theta - \cos^2 \theta)(1 - 2\sin^2 \theta \cos^2 \theta)$$

$$\begin{aligned} & \sin^8 \theta - \cos^8 \theta \\ &= (\sin^4 \theta)^2 - (\cos^4 \theta)^2 \\ &= (\sin^4 \theta - \cos^4 \theta)(\sin^4 \theta + \cos^4 \theta) \\ &= (\sin^2 \theta - \cos^2 \theta)(\sin^2 \theta + \cos^2 \theta)(\sin^4 \theta + \cos^4 \theta) \\ &= (\sin^2 \theta - \cos^2 \theta)(\sin^4 \theta + \cos^4 \theta) \\ &= (\sin^2 \theta - \cos^2 \theta)[(\sin^2 \theta)^2 + (\cos^2 \theta)^2 + 2\sin^2 \theta \cos^2 \theta - 2\sin^2 \theta \cos^2 \theta] \\ &= (\sin^2 \theta - \cos^2 \theta)[(\sin^2 \theta + \cos^2 \theta)^2 - 2\sin^2 \theta \cos^2 \theta] \\ &= (\sin^2 \theta - \cos^2 \theta)(1 - 2\sin^2 \theta \cos^2 \theta) \end{aligned}$$

$$(viii) \sec^4 A - \sec^2 A = \frac{\sin^2 A}{\cos^4 A}$$

$$\begin{aligned} \sec^4 A - \sec^2 A &= \frac{1}{\cos^4 A} - \frac{1}{\cos^2 A} \\ &= \frac{1 - \cos^2 A}{\cos^4 A} \\ &= \frac{\sin^2 A}{\cos^4 A} \quad [\because \sin^2 A = 1 - \cos^2 A] \end{aligned}$$

$$(ix) \frac{\tan^2 \theta}{\tan^2 \theta - 1} + \frac{\operatorname{cosec}^2 \theta}{\sec^2 \theta - \operatorname{cosec}^2 \theta} = \frac{1}{\sin^2 \theta - \cos^2 \theta}$$

$$\begin{aligned} \frac{\tan^2 \theta}{\tan^2 \theta - 1} + \frac{\operatorname{cosec}^2 \theta}{\sec^2 \theta - \operatorname{cosec}^2 \theta} &= \frac{\frac{\sin^2 \theta}{\cos^2 \theta}}{\frac{\sin^2 \theta}{\cos^2 \theta} - 1} + \frac{\frac{1}{\sin^2 \theta}}{\frac{1}{\cos^2 \theta} - \frac{1}{\sin^2 \theta}} \\ &= \frac{\sin^2 \theta}{\sin^2 \theta - \cos^2 \theta} + \frac{\frac{1}{\sin^2 \theta}}{\frac{\sin^2 \theta - \cos^2 \theta}{\cos^2 \theta \sin^2 \theta}} \\ &= \frac{\sin^2 \theta}{\sin^2 \theta - \cos^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} \\ &= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta} = \frac{1}{\sin^2 \theta - \cos^2 \theta} \quad (\because \sin^2 \theta + \cos^2 \theta = 1) \end{aligned}$$

$$(x) \frac{\sec^2 \theta - \sin^2 \theta}{\tan^2 \theta} = \operatorname{cosec}^2 \theta - \cos^2 \theta$$

$$\begin{aligned} \frac{\sec^2 \theta - \sin^2 \theta}{\tan^2 \theta} &= \frac{\frac{1}{\cos^2 \theta} - \sin^2 \theta}{\frac{\sin^2 \theta}{\cos^2 \theta}} \\ &= \frac{1 - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} \\ &= \frac{1 - \sin^2 \theta \cos^2 \theta}{\sin^2 \theta} \\ &= \frac{1}{\sin^2 \theta} - \frac{\sin^2 \theta \cos^2 \theta}{\sin^2 \theta} \\ &= \operatorname{cosec}^2 \theta - \cos^2 \theta \end{aligned}$$

$$\begin{aligned}
 \text{(xi)} \quad & \frac{\cos^3 \theta + \sin^3 \theta}{\cos \theta + \sin \theta} + \frac{\cos^3 \theta - \sin^3 \theta}{\cos \theta - \sin \theta} = 2 \\
 & \frac{\cos^3 \theta + \sin^3 \theta}{\cos \theta + \sin \theta} + \frac{\cos^3 \theta - \sin^3 \theta}{\cos \theta - \sin \theta} \\
 & = \frac{(\cos^3 \theta + \sin^3 \theta)(\cos \theta - \sin \theta) + (\cos^3 \theta - \sin^3 \theta)(\cos \theta + \sin \theta)}{(\cos \theta + \sin \theta)(\cos \theta - \sin \theta)} \\
 & = \frac{\cos^4 \theta - \cos^3 \theta \sin \theta + \sin^3 \theta \cos \theta - \sin^4 \theta + \cos^4 \theta + \cos^3 \theta \sin \theta - \sin^3 \theta \cos \theta - \sin^4 \theta}{\cos^2 \theta - \sin^2 \theta} \\
 & = \frac{2 \cos^4 \theta - 2 \sin^4 \theta}{\cos^2 \theta - \sin^2 \theta} = \frac{2(\cos^4 \theta - \sin^4 \theta)}{\cos^2 \theta - \sin^2 \theta} \\
 & = \frac{2(\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta)}{(\cos^2 \theta - \sin^2 \theta)} = 2(\cos^2 \theta + \sin^2 \theta) \\
 & = 2 \quad (\because \cos^2 \theta + \sin^2 \theta = 1)
 \end{aligned}$$

OR

$$\begin{aligned}
 & \frac{\cos^3 \theta + \sin^3 \theta}{\cos \theta + \sin \theta} + \frac{\cos^3 \theta - \sin^3 \theta}{\cos \theta - \sin \theta} \\
 & = \frac{(\cos \theta + \sin \theta)(\cos^2 \theta + \sin^2 \theta - \cos \theta \sin \theta)}{(\cos \theta + \sin \theta)} + \frac{(\cos \theta - \sin \theta)(\cos^2 \theta + \sin^2 \theta + \cos \theta \sin \theta)}{(\cos \theta - \sin \theta)} \\
 & (\because a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)) \\
 & = (\cos^2 \theta + \sin^2 \theta - \cos \theta \sin \theta) + (\cos^2 \theta + \sin^2 \theta + \cos \theta \sin \theta) \\
 & = 1 - \cos \theta \sin \theta + 1 + \cos \theta \sin \theta \quad (\because \cos^2 \theta + \sin^2 \theta = 1) \\
 & = 2
 \end{aligned}$$

$$\begin{aligned}
 \text{(xii)} \quad & \frac{\tan \theta + \sin \theta}{\tan \theta - \sin \theta} = \frac{\sec \theta + 1}{\sec \theta - 1} \\
 & \frac{\tan \theta + \sin \theta}{\tan \theta - \sin \theta} \\
 & = \frac{\frac{\sin \theta}{\cos \theta} + \sin \theta}{\frac{\sin \theta}{\cos \theta} - \sin \theta} = \frac{\sin \theta + \sin \theta \cos \theta}{\sin \theta + \sin \theta \cos \theta} \\
 & = \frac{\sin \theta(1 + \cos \theta)}{\sin \theta(1 + \cos \theta)} = \frac{1 + \cos \theta}{1 - \cos \theta} \\
 & = \frac{1 + \frac{1}{\sec \theta}}{1 - \frac{1}{\sec \theta}} = \frac{\frac{\sec \theta + 1}{\sec \theta}}{\frac{\sec \theta - 1}{\sec \theta}} = \frac{\sec \theta + 1}{\sec \theta - 1}
 \end{aligned}$$

$$\begin{aligned}
 \text{(xiii)} \quad & \left[\frac{1}{(\sec^2 \theta - \cos^2 \theta)} + \frac{1}{(\operatorname{cosec}^2 \theta - \sin^2 \theta)} \right] (\sin^2 \theta \cos^2 \theta) = \frac{1 - \sin^2 \theta \cos^2 \theta}{2 + \sin^2 \theta \cos^2 \theta} \\
 & \left[\frac{1}{(\sec^2 \theta - \cos^2 \theta)} + \frac{1}{(\operatorname{cosec}^2 \theta - \sin^2 \theta)} \right] (\sin^2 \theta \cos^2 \theta) \\
 & = \left[\frac{1}{\left(\frac{1}{\cos^2 \theta} - \cos^2 \theta \right)} + \frac{1}{\left(\frac{1}{\sin^2 \theta} - \sin^2 \theta \right)} \right] (\sin^2 \theta \cos^2 \theta) \\
 & = \left[\frac{1}{\left(\frac{1 - \cos^4 \theta}{\cos^2 \theta} \right)} + \frac{1}{\left(\frac{1 - \sin^4 \theta}{\sin^2 \theta} \right)} \right] (\sin^2 \theta \cos^2 \theta) \\
 & = \left[\frac{\cos^2 \theta}{1 - \cos^4 \theta} + \frac{\sin^2 \theta}{1 - \sin^4 \theta} \right] (\sin^2 \theta \cos^2 \theta) \\
 & = \left[\frac{\cos^2 \theta - \cos^2 \theta \sin^2 \theta + \sin^2 \theta - \sin^2 \theta \cos^2 \theta}{(1 - \cos^4 \theta)(1 - \sin^4 \theta)} \right] (\sin^2 \theta \cos^2 \theta) \\
 & = \left[\frac{\cos^2 \theta + \sin^2 \theta - \cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta)}{(1 - \cos^2 \theta)(1 + \cos^2 \theta)(1 - \sin^2 \theta)(1 + \sin^2 \theta)} \right] (\sin^2 \theta \cos^2 \theta) \\
 & = \left[\frac{1 - \cos^2 \theta \sin^2 \theta}{\sin^2 \theta (1 + \cos^2 \theta) \cos^2 \theta (1 + \sin^2 \theta)} \right] (\sin^2 \theta \cos^2 \theta) \\
 & (\because \cos^2 \theta + \sin^2 \theta = 1, (1 - \cos^2 \theta) = \sin^2 \theta, (1 - \sin^2 \theta) = \cos^2 \theta) \\
 & = \frac{1 - \cos^2 \theta \sin^2 \theta}{(1 + \cos^2 \theta)(1 + \sin^2 \theta)} = \frac{1 - \cos^2 \theta \sin^2 \theta}{1 + \sin^2 \theta + \cos^2 \theta + \sin^2 \theta \cos^2 \theta} \\
 & = \frac{1 - \cos^2 \theta \sin^2 \theta}{1 + 1 + \sin^2 \theta \cos^2 \theta} = \frac{1 - \sin^2 \theta \cos^2 \theta}{2 + \sin^2 \theta \cos^2 \theta}
 \end{aligned}$$

$$\begin{aligned}
\text{(xiv)} \quad & \frac{\cot^2 \theta (\sec \theta - 1)}{(1 + \sin \theta)} = \sec^2 \theta \left(\frac{1 - \sin \theta}{1 + \sec \theta} \right) \\
& \frac{\cot^2 \theta (\sec \theta - 1)}{(1 + \sin \theta)} \\
& = \frac{\cot^2 \theta (\sec \theta - 1)(1 - \sin \theta)(\sec \theta + 1)}{(1 + \sin \theta)(1 - \sin \theta)(\sec \theta + 1)} \\
& = \frac{\cot^2 \theta (\sec \theta - 1)(\sec \theta + 1)(1 - \sin \theta)}{(1 + \sin \theta)(1 - \sin \theta)(\sec \theta + 1)} \\
& = \frac{\cot^2 \theta (\sec^2 \theta - 1)(1 - \sin \theta)}{(1 - \sin^2 \theta)(1 + \sec \theta)} \\
& = \frac{\cot^2 \theta (\tan^2 \theta)(1 - \sin \theta)}{(\cos^2 \theta)(1 + \sec \theta)} \quad (\because \tan^2 \theta = \sec^2 \theta - 1, 1 - \sin^2 \theta = \cos^2 \theta) \\
& = \frac{(\cot \theta \tan \theta)^2 (1 - \sin \theta)}{(\cos^2 \theta)(1 + \sec \theta)} \\
& = \frac{1(1 - \sin \theta)}{(\cos^2 \theta)(1 + \sec \theta)} \quad (\because \cot \theta \tan \theta = 1) \\
& = \sec^2 \theta \left(\frac{1 - \sin \theta}{1 + \sec \theta} \right)
\end{aligned}$$